Finite Sample Spaces

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Finite Sample Space

We now deal exclusively with experiments for which the sample space S consists of a *finite number* of elements. That is, we suppose that S may be written as $S = \{a_1, a_2, \ldots, a_k\}$.

In order to characterize $P(A)$ for this model we shall first consider the event consisting of a *single outcome*, sometimes called an **elementary event**, say $A = \{a_i\}$. We proceed as follows.

To each elementary event $\{a_i\}$ we assign a number p_i, called the probability of $\{a_i\}$, satisfying the following conditions:

(a) $p_i \geq 0, \quad i = 1, 2, \ldots, k,$

(b) $p_1 + p_2 + \cdots + p_k = 1.$

Finite Sample Space

Next, suppose that an event A consists of r outcomes, $1 \leq r \leq k$, say

$$A = \{a_{j_1}, a_{j_2}, \ldots, a_{j_r}\},$$

where $j_1, j_2, \ldots, j_r$ represent any r indices from $1, 2, \ldots, k$. Hence it follows that

$$P(A) = p_{j_1} + p_{j_2} + \cdots + p_{j_r}. \tag{1}$$

To summarize: The assignment of probabilities p_i to each elementary event $\{a_i\}$, subject to the conditions (a) and (b) above, uniquely determines $P(A)$ for each event $A \subset S$, where $P(A)$ is given by Equation (1).

In order to evaluate the individual p_j's, some assumption concerning the individual outcomes must be made.

Example

Example 1.

Suppose that only three outcomes are possible in an experiment, say a_1, a_2, *and* a_3. *Suppose furthermore, that* a_1 *is twice as probable to occur as* a_2, *which in turn is twice as probable to occur as* a_3.

Hence $p_1 = 2p_2$ *and* $p_2 = 2p_3$. *Since* $p_1 + p_2 + p_3 = 1$, *we have* $4p_3 + 2p_3 + p_3 = 1$, *which finally yields*

$$p_3 = \frac{1}{7} \quad p_2 = \frac{2}{7}, \quad \text{and} \quad p_1 = \frac{4}{7}.$$

We shall use the phrase "equally likely" to mean "equally probable".

Equally Likely Outcomes

The most commonly made assumption for finite sample spaces is that **all outcomes are equally likely**. This assumption can by no means be taken for granted, however; it must be carefully justified. There are many experiments for which such an assumption is warranted, but there are also many experimental situations in which it would be quite erroneous to make this assumption.

For example, it would be quite unrealistic to suppose that it is as likely for no telephone calls to come into a telephone exchange between 1 am and 2 am as between 5 pm and 6 pm.

If all the k outcomes are equally likely, it follows that each $p_i = 1/k$. For the condition $p_1 + \cdots + p_k = 1$ becomes $kp_i = 1$ for all i. From this it follows that for any event A consisting of r outcomes, we have

$$P(A) = r/k.$$

Equally Likely Outcomes

This method of evaluating $P(A)$ is often stated as follows:

$$P(A) = \frac{\text{number of ways in which } E \text{ can occur favorable to } A}{\text{total number of ways in which } E \text{ can occur}}.$$

It is important to realize that the above expression for $P(A)$ is only a consequence of the assumption that all outcomes are equally likely and is only applicable when this assumption is fulfilled. It most certainly does not serve as a general definition of probability.

Example 2.

A die is tossed and all outcomes are assumed to be equally likely. The event A occurs if and only if a number larger than 4 shows. That is, $A = \{5, 6\}$. Hence $P(A) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$.

Example

Example 3.

An honest coin is tossed two times. Let A be the event: $\{$one head appears$\}$. In evaluating $P(A)$ one analysis of the problem might be as follows. The sample space is $S = \{0, 1, 2\}$, where each outcome represents the number of heads that occur. Hence $P(A) = \frac{1}{3}$! This analysis is obviously incorrect, since for the sample space considered above, all outcomes are not equally likely. In order to apply the above methods we should consider, instead, the sample space $S' = \{HH, HT, TH, TT\}$. In this sample space all outcomes are equally likely, and hence we obtain for the correct solution of our problem, $P(A) = \frac{2}{4} = \frac{1}{2}$. We could use the sample space S correctly as follows: The outcomes 0 and 2 are equally likely, while the outcome 1 is twice as likely as either of the others. Hence $P(A) = \frac{1}{2}$, which checks with the above answer.

Example

This example illustrates two points. First, we must be quite sure that all outcomes may be assumed to be equally likely before using the above procedure.

Second, we can often, by an appropriate choice of the sample space, reduce the problem to one in which all outcomes are equally likely. Whenever possible this should be done, since it usually makes the computation simpler.

Quite often the manner in which the experiment is executed determines whether or not the possible outcomes are equally likely.

Example

For instance, suppose that we choose one bolt from a box containing three bolts of different sizes. If we choose the bolt by simply reaching into the box and picking the one which we happen to touch first, it is obvious that the largest bolt will have a greater probability of being chosen than the other two.

However, by carefully labeling each bolt with a number, writing the number on a tag, and choosing a tag, we can try to ensure that each bolt has in fact the same probability of being chosen. Thus we may have to go to considerable trouble in order to ensure that the mathematical assumption of equally likely outcomes is in fact appropriate.

Remarks

Suppose that we have N objects, say $a_1, a_2, \dots, a_N$.

1. *To choose one object at random* from the N objects means that each object has the same probability of being chosen. That is, Prob (chooing a_i) $= 1/N, \quad i = 1, 2, \dots, N$.
2. *To choose two objects at random* from N objects means that each pair of objects (disregarding order) has the same probability of being chosen as any other pair. For example, if we must choose two objects at random from (a_1, a_2, a_3, a_4), then obtaining a_1 and a_2 is just as probable as obtaining a_2 and a_3, etc. This formulation immediately raises the question of *how many different* pairs there are. For suppose that there are K such pairs. Then the probability of each pair would be $1/K$. We shall learn shortly how to compute K.
3. *To choose n objects at random* $(n \leq N)$ from the N objects means that each n-tuple, say $a_{i_1}, a_{i_2}, \dots, a_{i_n}$, is as likely to be chosen as any other n-tuple.

Communication System

We consider the following problem in communication system to find the probability for the system to be functional.

A communication system is to consist of n seemingly identical antennas that are to be lined up in a linear order. The resulting system will then be able to receive all incoming signals – and will be called *functional* – as long as no two consecutive antennas are defective. If it turns out that exactly m of the n antennas are defective, what is the probability that the resulting system will be functional?

Communication System

For instance, in the special case where $n = 4$ and $m = 2$, there are 6 possible system configurations, namely,

$$\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array}$$

where 1 means that the antenna is working and 0 that it is defective.

Communication System

Because the resulting system will be functional in the first 3 arrangements and not functional in the remaining 3, it seems reasonable to take $\frac{3}{6} = \frac{1}{2}$ as the desired probability.

In the case of general n and m, we could compute the probability that the system is functional in a similar fashion. That is, we could count the number of configurations that result in the system's being functional and then divide by the total number of all possible configurations.

Many problems in probability theory can be solved simply by counting the number of different ways that a certain event can occur. The mathematical theory of counting is formally known as *combinatorial analysis*.

The basic principle of counting

We now discuss some principles of counting.

Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

Example

Example 4.

A small community consists of 10 *women, each of whom has* 3 *children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?*

Solution. We see from the basic principle that there are $10 \times 3 = 30$ possible choices.

The generalized basic principle of counting

When there are more than two experiments to be performed, the basic principle can be generalized.

If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and if $\dots$, then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

Example 5.

A college planning committee consists of 3 *freshmen,* 4 *sophomores,* 5 *juniors, and* 2 *seniors. A subcommittee of* 4*, consisting of* 1 *person from each class, is to be chosen. How many different subcommittees are possible?*

Solution. It follows from the generalized version of the basic principle that there are $3 \times 4 \times 5 \times 2 = 120$ possible subcommittees.

Example 6.

How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Solution. The answer is $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175,760,000$.

Example 7.

How many functions defined on n points are possible if each functional value is either 0 *or* 1*?*

Solution. There are 2^n possible functions.

Example 8.

In the above example how many license plates would be possible if repetition among letters or numbers were prohibited?

Solution. In this case, there would be
$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 78,624,000$ possible license plates.

Permutations

How many different ordered arrangements of the letters a, b, and c are possible? By direct enumeration we see that there are 6, namely, abc, acb, bac, bca, cab, and cba. Each arrangement is known as a *permutation*. Thus, there are 6 possible permutations of a set of 3 objects. This result could also have been obtained from the basic principle, since the first object in the permutation can be any of the 3, the second object in the permutation can then be chosen from any of the remaining 2, and the third object in the permutation is then the remaining 1. Thus, there are $3 \cdot 2 \cdot 1 = 6$ possible permutations.

Suppose now that we have n objects. Reasoning similar to that we have just used for the 3 letters then shows that there are

$$n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

different permutations of the n objects.

Example

Example 9.

How many different batting orders are possible for a baseball team consisting of 9 *players?*

Solution. There are $9! = 362,880$ possible batting orders.

Example 10.

A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.

1. *How many different rankings are possible?*
2. *If the men are ranked just among themselves and the women just among themselves, how many different rankings are possible?*

Solution

1. Because each ranking corresponds to a particular ordered arrangement of the 10 people, the answer to this part is $10! = 3,628,800$.
2. Since there are $6!$ possible rankings of the men among themselves and $4!$ possible rankings of the women among themselves, it follows from the basic principle that there are $(6!)(4!) = (720)(24) = 17,280$ possible rankings in this case.

Example 11.

Ms. Jones has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Ms. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

Solution. There are $4!\ 3!\ 2!\ 1!$ arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Similarly, for each possible ordering of the subjects, there are $4!\ 3!\ 2!\ 1!$ possible arrangements. Hence, as there are $4!$ possible orderings of the subjects, the desired answer is

$4!\ 4!\ 3!\ 2!\ 1! = 6912$

Example

We shall now determine the number of permutations of a set of n objects when certain of the objects are indistinguishable from each other.

Example 12.

How many different letter arrangements can be formed from the letters PEPPER?

There are $6!/(3!\ 2!) = 60$ possible letter arrangements of the letters PEPPER.

Example

In general, the same reasoning shows that there are

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, $\ldots, n_r$ are alike.

Example 13.

A chess tournament has 10 competitors, of which 4 are Russian, 3 are from the United States, 2 are from Great Britain, and 1 is from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?

Solution. There are

$$\frac{10!}{4!\ 3!\ 2!\ 1!} = 12,600$$

possible outcomes.

Example 14.

How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?

Solution. There are

$$\frac{9!}{4!\,3!\,2!} = 1260$$

different signals.

Combinations

We are often interested in determining the number of different groups of r objects that could be formed from a total of n objects. For instance, how many different groups of 3 could be selected from the 5 items A, B, C, D, and E? To answer this question, reason as follows: Since there are 5 ways to select the initial item, 4 ways to then select the next item, and 3 ways to select the final item, there are thus $5 \cdot 4 \cdot 3$ ways of selecting the group of 3 **when the order in which the items are selected** is relevant.

In general, as $n(n-1)\cdots(n-r+1)$ represents the number of different ways that a group of r items could be selected from n items when the order of selection is relevant, and as each group of r items will be counted $r!$ times in this count, it follows that the number of different groups of r items that could be formed from a set of n items is

$$\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!\ r!}.$$

Notation and terminology

We define $\binom{n}{r}$, for $r \leq n$, by

$$\binom{n}{r} = \frac{n!}{(n-r)!\ r!}$$

and say that $\binom{n}{r}$ represents the number of possible combinations of n objects taken r at a time [1].

Thus, $\binom{n}{r}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of **selection is not considered relevant**.

[1] By convention, 0! is defined to be 1. Thus, $\binom{n}{0} = \binom{n}{n} = 1$. We also take $\binom{n}{i}$ to be equal to 0 when either $i < 0$ or $i > n$.

Example

Example 15.

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Answer: 1140 possible committees.

Example 16.

From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together?

Answer : 300 possible committees.

Example 17.

Consider a set of n antennas of which m are defective and $n - m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

Solution : There are $\binom{n-m+1}{m}$ possible orderings in which there is at least one functional antenna between any two defective ones.

A useful combinatorial identity is

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad 1 \leq r \leq n.$$

The binomial theorem

The values $\binom{n}{r}$ are often referred to as *binomial coefficients* because of their prominence in the binomial theorem.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example 18.

How many subsets are there of a set consisting of n elements?

Solution: Since there are $\binom{n}{k}$ subsets of size k, the desired answer is

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n.$$

Multinomial Coefficients

A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. How many different divisions are possible?

To answer this question, we note that there are $\binom{n}{n_1}$ possible choices for the first group; for each choice of the first group, there are $\binom{n-n_1}{n_2}$ possible choices for the second group; for each choice of the first two groups, there are $\binom{n-n_1-n_2}{n_3}$ possible choices for the third group; and so on.

Multinomial Coefficients

It then follows from the generalized version of the basic counting principle that there are

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$$
$$=\frac{n!}{(n-n_1)!n_1!}\frac{(n-n_1)!}{(n-n_1-n_2)!n_2!}\cdots\frac{(n-n_1-n_2-\cdots-n_{r-1})!}{0!n_r!}$$
$$=\frac{n!}{n_1!n_2!\cdots n_r!}$$

possible divisions.

Multinomial Coefficients

If $n_1 + n_2 + \cdots + n_r = n$, we define $\binom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}.$$

Thus, $\binom{n}{n_1, n_2, \ldots, n_r}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

Example 19.

A police department in a small city consists of 10 *officers. If the department policy is to have* 5 *of the officers patrolling the streets,* 2 *of the officers working full time at the station, and* 3 *of the officers on reserve at the station, how many different divisions of the* 10 *officers into the* 3 *groups are possible?*

Solution: 2520 possible divisions.

Example 20.

Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

Solution: There are $\frac{10!}{5!5!} = 252$ possible divisions.

Example 21.

In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

Solution: The desired answer is

$$\frac{10!/(5!5!)}{2!} = 126.$$

The Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1,\ldots,n_r): \\ n_1+\cdots+n_r=n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

That is, the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$.

The numbers $\binom{n}{n_1, n_2, \ldots, n_r}$ are known as *multinomial coefficients*.

Example

Example 22.

In the first round of a knockout tournament involving $n = 2^m$ players, the n players are divided into $n/2$ pairs, with each of these pairs then playing a game. The losers of the games are eliminated while the winners go on to the next round, where the process is repeated until only a single player remains. Suppose we have a knockout tournament of 8 players.

(a) *How many possible outcomes are there for the initial round? (For instance, one outcome is that 1 beats 2, 3 beats 4, 5 beats 6, and 7 beats 8.)*

(b) *How many outcomes of the tournament are possible, where an outcome gives complete information for all rounds?*

Example

Example 23.

$$\begin{aligned}(x_1+x_2+x_3)^2 &= \binom{2}{2,0,0} x_1^2 x_2^0 x_3^0 + \binom{2}{0,2,0} x_1^0 x_2^2 x_3^0 \\ &+ \binom{2}{0,0,2} x_1^0 x_2^0 x_3^2 + \binom{2}{1,1,0} x_1^1 x_2^1 x_3^0 \\ &+ \binom{2}{1,0,1} x_1^1 x_2^0 x_3^1 + \binom{2}{0,1,1} x_1^0 x_2^1 x_3^1 \\ &= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3\end{aligned}$$

The Number of Integer Solutions of Equations

There are r^n possible outcomes when n distinguishable balls are to be distributed into r distinguishable urns. This result follows because each ball may be distributed into any of r possible urns. Let us now, however, suppose that the n balls are indistinguishable from each other. In this case, how many different outcomes are possible? As the balls are indistinguishable, it follows that the outcome of the experiment of distributing the n balls into r urns can be described by a vector $(x_1, x_2, \ldots, x_r)$, where x_i denotes the number of balls that are distributed into the ith urn. Hence, the problem reduces to finding the number of distinct nonnegative integer-valued vectors $(x_1, x_2, \ldots, x_r)$ such that

$$x_1 + x_2 + \cdots + x_r = n.$$

Multinomial Coefficients

Proposition 24.

There are $\binom{n-1}{r-1}$ *distinct positive integer-valued vectors* $(x_1, x_2, \ldots, x_r)$ *satisfying the equation*

$$x_1 + x_2 + \cdots + x_r = n \quad x_i > 0, i = 1, \ldots, r.$$

To obtain the number of nonnegative (as opposed to positive) solutions, note that the number of nonnegative solutions of $x_1 + x_2 + \cdots + x_r = n$ *is the same as the number of positive solutions of* $y_1 + \cdots + y_r = n + r$ *(seen by letting* $y_i = x_i + 1$, $i = 1, \ldots, r$*).*

Proposition 25.

There are $\binom{n+r-1}{r-1}$ *distinct nonnegative integer-valued vectors* $(x_1, x_2, \ldots, x_r)$ *satisfying the equation*

$$x_1 + x_2 + \cdots + x_r = n.$$

Example

Example 26.

How many distinct nonnegative integer-valued solutions of $x_1 + x_2 = 3$ are possible?

Solution: There are $\binom{3+2-1}{2-1} = 4$ such solutions: $(0,3),(1,2),(2,1),(3,0)$.

Example 27.

An investor has 20 thousand dollars to invest among 4 possible investments. Each investment must be in units of a thousand dollars. If the total 20 thousand is to be invested, how many different investment strategies are possible? What if not all the money need be invested?

Solution: If we let x_i, $i = 1, 2, 3, 4$, denote the number of thousands invested in investment i, then, when all is to be invested, x_1, x_2, x_3, x_4 are integers satisfying the equation

$$x_1 + x_2 + x_3 + x_4 = 20 \quad x_i \geq 0.$$

Hence there are $\binom{23}{3} = 1771$ possible investment strategies. If not all of the money need be invested, then if we let x_5 denote the amount kept in reserve, a strategy is a nonnegative integer-valued vector $(x_1, x_2, x_3, x_4, x_5)$ satisfying the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20.$$

Hence there are now $\binom{24}{4} = 10,626$ possible strategies.

Example 28.

How many terms are there in the multinomial expansion of $(x_1 + x_2 + \cdots + x_r)^n$*?*

Solution:

$$(x_1 + x_2 + \cdots + x_r)^n = \sum \binom{n}{n_1, \ldots, n_r} x_1^{n_1} \cdots x_r^{n_r}$$

where the sum is over all nonnegative integer-valued $(n_1, \ldots, n_r)$ such that $n_1 + \cdots + n_r = n$. Hence there are $\binom{n+r-1}{r-1}$ such terms.

Example 29.

Let us consider a set of n antennas of which we have a set of n items, of which m are (indistinguishable and) defective and the remaining $n - m$ *are (also indistinguishable and) functional. Our objective is to determine the number of linear orderings in which no two defectives are next to each other. To determine this number, let us imagine that the defective items are lined up among themselves and the functional ones are now to be put in position. Let us denote* x_1 *as the number of functional items to the left of the first defective,* x_2 *as the number of functional items between the first two defectives, and so on. That is, schematically, we have*

$$x_1 0 x_2 0 \cdots x_m 0 x_{m+1}.$$

Example

Now, there will be at least one functional item between any pair of defectives as long as $x_i > 0$, $i = 2, \ldots, m$. Hence, the number of outcomes satisfying the condition is the number of vectors $x_1, \ldots, x_{m+1}$ that satisfy the equation

$$x_1 + \cdots + x_{m+1} = n - m \quad x_1 \geq 0,\, x_{m+1} \geq 0,\, x_i > 0,\, i = 2, \ldots, m.$$

But, on letting $y_1 = x_1 + 1, y_i = x_i, i = 2, \ldots, m, y_{m+1} = x_{m+1} + 1$, we see that this number is equal to the number of positive vectors $(y_1, \ldots, y_{m+1})$ that satisfy the equation

$$y_1 + y_2 + \cdots + y_{m+1} = n - m + 2.$$

Hence there are $\binom{n-m+1}{m}$ such outcomes.

Suppose now that we are interested in the number of outcomes in which each pair of defective items is separated by at least 2 functional items. By the same reasoning as that applied previously, this would equal the number of vectors satisfying the equation

$$x_1 + \cdots + x_{m+1} = n - m \quad x_1 \geq 0,\, x_{m+1} \geq 0,\, x_i \geq 2,\, i = 2, \ldots, m.$$

Upon letting $y_1 = x_1 + 1,\ y_i = x_i - 1,\ i = 2, \ldots, m,\ y_{m+1} = x_{m+1} + 1$, we see that this is the same as the number of positive solutions of the equation

$$y_1 + \cdots + y_{m+1} = n - 2m + 3.$$

Hence there are $\binom{n-2m+2}{m}$ such outcomes.

Multiplication Principle

Suppose that a procedure, designated by 1 can be performed in n_1 ways. Let us assume that a second procedure, designated by 2, can be performed in n_2 ways. Suppose also that each way of doing 1 may be followed by any way of doing 2. Then the procedure consisting of 1 followed by 2 may be performed in $n_1 n_2$ ways.

This principle obviously may be extended to any number of procedures. If there are k procedures and the ith procedure may be performed in n_i ways, $i = 1, 2, \ldots k$, then the procedure consisting of 1, followed by $2, \ldots$ followed by procedure k may be performed in $n_1 n_2 \cdots n_k$ ways.

Example 30.

A manufactured item must pass through three control stations. At each station the item is inspected for a particular characteristic and marked accordingly. At the first station, three ratings are possible while at the last two stations four ratings are possible. Hence there are $3 \cdot 4 \cdot 4 = 48$ *ways in which the item may be marked.*

Addition Principle

Suppose that a procedure, designated by 1, can be performed in n_1 ways. Assume that a second procedure, designated by 2, can be performed in n_2 ways. Suppose **furthermore that it is not possible that both** 1 **and** 2 **are performed together.** Then the number of ways in which we can perform 1 or 2 is $n_1 + n_2$.

This principle, too, may be generalized as follows. If there are k procedures and the ith procedure may be performed in n_i ways, $i = 1, 2, \ldots, k$, then the number of ways in which we may perform procedure 1 or procedure 2 or $\cdots$ or procedure k is given by $n_1 + n_2 + \cdots + n_k$, assuming that no two procedures may be performed together.

Example 31.

Suppose that we are planning a trip and are deciding between bus or train transportation. If there are three bus routes and two train routes, then there are $3 + 2 = 5$ *different routes available for the trip.*

Permutations

(a) Suppose that we have n different objects. In how many ways, say nP_n, may these objects be arranged (permuted)? For example, if we have objects a, b, and c, we can consider the following arrangements: abc, acb, bac, bca, cab, and cba. Thus the answer is 6. In general, consider the following scheme. Arranging the n objects is equivalent to putting them into a box with n compartments, in some specified order.

1	2	·	·	·	n

The first slot may be filled in any one of n ways. the second slot in any one of $(n-1)$ ways, ..., and the last slot in exactly one way. Hence applying the above multiplication principle, we see that the box may be filled in $n(n-l)(n-2)\ldots 1$ ways.

Definition 32.

If n is a positive integer, we define $n! = (n)(n-1)(n-2)\cdots 1$ *and call it n-factorial. We also define* $0! = 1$.

Thus the number of permutations of n different objects is given by

$$nP_n = n!$$

(b) Consider again n different objects. This time we wish to choose r of these objects, $0 \le r \le n$, and permute the chosen r. We denote the number of ways of doing this by nP_r. We again resort to the above scheme of filling a box having n compartments; this time we simply stop after the rth compartment has been filled. Thus the first compartment may be filled in n ways, the second in $(n-1)$ ways, ... , and the rth compartment in $n-(r-1)$ ways.

Thus the entire procedure may be accomplished, again using the multiplication principle, in

$$n(n-1)(n-2)\cdots(n-r+1)$$

ways. Using the factorial notation introduced above, we may write

$$nP_r = \frac{n!}{(n-r)!}.$$

Combinations

Consider again n different objects. This time we are concerned with counting the number of ways we may choose r out of these n objects without regard to order. For example, we have the objects a, b, c, and d, and $r = 2$; we wish to count ab, ac, ad, bc, bd, and cd. In other words, we do not count ab and ba since the same objects are involved and only the order differs.

To obtain the general result we recall the formula derived above: The number of ways of choosing r objects out of n and permuting the chosen r equals $n!/(n-r)!$ Let C be the number of ways of choosing r out of n, disregarding order. (That is, C is the number sought.) Note that once the r items have been chosen, there are $r!$ ways of permuting them.

Hence applying the multiplication principle again, together with the above result, we obtain

$$Cr! = \frac{n!}{(n-r)!}.$$

Thus the number of ways of choosing r out of n different objects, disregarding order, is given by

$$C = \frac{n!}{r!(n-r)!}.$$

This number arises in many contexts in mathematics and hence a special symbol is used for it. We shall write

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

Combinations

For the present purpose $\binom{n}{r}$ is defined only if n is a positive integer and if r is an integer $0 \leq r \leq n$. However, we can define $\binom{n}{r}$ quite generally for any real number n and for any nonnegative integer r as follows:

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

The numbers $\binom{n}{r}$ often are called *binomial coefficients,* for they appear as coefficients in the expansion of the binomial expression $(a+b)^n$. If n is a positive integer, $(a+b)^n = (a+b)(a+b)\cdots(a+b)$. When multiplied out, each term will consist of the product of k a's and $(n-k)b$'s, $k = 0, 1, 2, \ldots, n$.

Combinations

How many terms will there be of the form $a^k b^{n-k}$? We simply count the number of ways in which we can choose k out of n a's, disregarding order. But this is precisely given by $\binom{n}{k}$. Hence we have what is known as the binomial theorem.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

The numbers $\binom{n}{r}$ have many interesting properties only two of which we mention here. (Unless otherwise stated, we assume n to be a positive integer and r an integer, $0 \leq r \leq n$.)

(a) $\binom{n}{r} = \binom{n}{n-r}$,

(b) $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

Combinations

In the above context the binomial coefficients $\binom{n}{k}$ are meaningful only if n and k are nonnegative integers with $0 \leq k \leq n$. However, if we write

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

we observe that the latter expression is meaningful if n is any real number and k is any nonnegative integer. Thus,

$$\binom{-3}{5} = \frac{(-3)(-4)\cdots(-7)}{5!},$$

and so on.

Combinations

Using this extended version of the binomial coefficients we can state the generalized *form of the binomial theorem:*

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{x} x^k$$

This series is meaningful **for any real** n **and for all** x **such that** $|x| < 1$. Observe that if n is a positive integer, the infinite series reduces to a finite number of terms since in that case $\binom{n}{k} = 0$ if $k > n$.

Example 33.

1. *From eight persons, how many committees of three members may be chosen? Since two committees are the same if they are made up of the same members (regardless of the order in which they were chosen), we have* $\binom{8}{3} = 56$ *possible committees.*
2. *From eight different flags, how many signals made up of three flags may be obtained? This problem seems very much like the one above. However, here order does make a difference, and hence we obtain* $8!/5! = 336$ *signals.*
3. *A group of eight persons consists of five men and three women. How many committees of three may be formed consisting of exactly two men? Here we must do two things: choose two men (out of five) and choose one woman (out of three). Hence we obtain for the required number* $\binom{5}{2} \cdot \binom{3}{1} = 30$ *committees.*

Examples (Contd...)

4. We can now verify a statement made earlier, namely that the number of subsets of a set having n members is 2^n (counting the empty set and the set itself). Simply label each member with a one or a zero, depending on whether the member is to be included in, or excluded from, the subset. There are two ways of labeling each member, and there are n such members. Hence the multiplication principle tells us that there are $2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2 = 2^n$ possible labelings. But each particular labeling represents a choice of a subset. For example, $(1, 1, 0, 0, 0, \ldots, 0)$ would consist of the subset made up of just a_1 and a_2. Again, $(1, 1, \ldots 1)$ would represent S itself, and $(0, 0, \ldots, 0)$ would represent the empty set.
5. We can obtain the above result by using the Addition principle as follows. In obtaining subsets we must choose the empty set, those subsets consisting of exactly one element, those consisting of exactly 2 elements, . . . , and the set itself consisting of all n elements. This may be done in

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

ways. However the sum of these binomial coefficients is simply the expansion of $(1+1)^n = 2^n$.

Example (Contd...)

From a lot consisting of 20 defective and 80 nondefective items we chose 10 at random (without replacement). The number of ways of doing this is $\binom{100}{10}$. Hence the probability of finding exactly 5 defective and 5 nondefective items among the chosen 10 is given by $\frac{\binom{20}{5}\binom{80}{5}}{\binom{100}{10}}$. By means of logarithms of factorials (which are tabulated) the above can be evaluated and equals 0.021.

Hypergeometric Probability

Example 34.

Let us generalize the above problem. Suppose that we have N items. If we choose n of these at random, without replacement, there are $\binom{N}{n}$ different possible samples, all of which have the same probability of being chosen. If the N items are made up of r_1 $A's$ and r_2 $B's$ (with $r_1 + r_2 = N$), then the probability that the n chosen items contain exactly s_1 $A's$ and $(n - s_1)$ $B's$ is given b y

$$\frac{\binom{r_1}{s_1}\binom{r_2}{n - s_1}}{\binom{N}{n}}.$$

The above is called a *hypergeometric probability.*

Remark

It is very important to specify, when we speak of choosing items at random, whether we choose *with or without replacement*. In most realistic descriptions we intend the latter. For instance, when we inspect a number of manufactured articles in order to discover how many defectives there might be, we usually do not intend to inspect the same item twice.

We have noted previously that the number of ways of choosing r things out of n, disregarding order, is given by $\binom{n}{r}$. The number of ways of choosing r things out of n, with replacement, is given by n^r. Here we are concerned about the order in which the items were chosen.

Example 35.

Suppose that we choose two objects at random from the four objects labeled a, b, c, and d.

1. *If we choose without replacement, the sample space S may be represented as follows:*

$$S = \{(a, b); (a, c); (b, c); (b, d); (c, d); (a, d)\}.$$

   There are $\binom{4}{2} = 6$ *possible outcomes. Each individual outcome indicates only which two objects were chosen and not the order in which they were chosen.*

2. *If we choose with replacement, the sample space S' may be represented as follows:*

$$\left\{ \begin{array}{l} (a, a); (a, b); (ac); (a, d); (b, a); (b, b); (b, c); (b, d); \\ (c, a); (c, b); (c, c); (c, d); (d, a); (d, b); (d, c); (d, d) \end{array} \right\}.$$

There are $4^2 = 16$ *possible outcomes. Here each individual outcome indicates which objects were chosen and the order in which they were chosen. Choosing at random implies that if we choose without replacement, all the outcomes in S are equally likely, while if we choose with replacement, then all the outcomes in S' are equally likely. Thus, if A is the event {the object c is chosen}, then we have from* $S, P(A) = \frac{3}{6} = \frac{1}{2}$ *if we choose without replacement, and from* $S', P(A) = \frac{7}{16}$ *if we choose with replacement.*

Permutations when not all objects are different.

In all the enumeration methods introduced we have assumed that all the objects under consideration were different (that is, distinguishable). However, this is not always the case.

Suppose, then, that we have n objects such that there are n_1 of one kind, n_2 of a second kind, ..., n_k of a kth kind, where $n_1 + n_2 + \cdot \quad + n_k = n$. Then the number of permutations of these n objects is given by

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Note that if all the objects are different, we have $n_i = 1, i = 1, 2, \ldots, k$, and hence the above formula reduces to $n!$, the previously obtained result.

Exercise 36.

How many outcome sequences are possible when a die is rolled four times, where we say, for instance, that the outcome is $3, 4, 3, 1$ *if the first roll landed on* 3*, the second on* 4*, the third on* 3*, and the fourth on* 1*?*

Solution: $6^4 = 1296$

Exercise 37.

Twenty workers are to be assigned to 20 different jobs, one to each job. How many different assignments are possible?

Solution: An assignment is a sequence $i_1, \ldots, i_{20}$ where i_j is the job to which person j is assigned. Since only one person can be assigned to a job, it follows that the sequence is a permutation of the numbers $1, \ldots, 20$ and so there are 20! different possible assignments.

Exercise 38.

In how many ways can 8 people be seated in a row if

(a) *there are no restrictions on the seating arrangement?*

(b) *persons A and B must sit next to each other?*

(c) *there are 4 men and 4 women and no 2 men or 2 women can sit next to each other?*

(d) *there are 5 men and they must sit next to each other?*

(e) *there are 4 married couples and each couple must sit together?*

Solution:

(a) $8! = 40,320$

(b) $2 \cdot 7! = 10,080$

(d) $5!4! = 2,880$

(e) $4!2^4 = 384$

Exercise 39.

In how many ways can 3 novels, 2 mathematics books, and 1 chemistry book be arranged on a bookshelf if

(a) *the books can be arranged in any order?*

(b) *the mathematics books must be together and the novels must be together?*

(c) *the novels must be together, but the other books can be arranged in any order?*

Solution:

(a) $6!$

(b) $3!2!3!$

(c) $3!4!$

Exercise 40.

Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to selected students from a class of 30. How many different outcomes are possible if

(a) *a student can receive any number of awards?*

(b) *each student can receive at most 1 award?*

Solution:

(a) 30^5

(b) $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26$

Exercise 41.

Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

Solution: $\binom{20}{2}$

Exercise 42.

From a group of 8 women and 6 men, a committee consisting of 3 men and 3 women is to be formed.
How many different committees are possible if

(a) *2 of the men refuse to serve together?*

(b) *2 of the women refuse to serve together?*

(c) *1 man and 1 woman refuse to serve together?*

Exercise

Solution:

(a) There are $\binom{8}{3}\binom{4}{3}+\binom{8}{3}\binom{2}{1}\binom{4}{2}=896$ possible committees.

There are $\binom{8}{3}\binom{4}{3}$ that do not contain either of the 2 men, and there are $\binom{8}{3}\binom{2}{1}\binom{4}{2}$ that contain exactly 1 of them.

(b) There are $\binom{6}{3}\binom{6}{3}+\binom{2}{1}\binom{6}{2}\binom{6}{3}=1000$ possible committees.

(c) There are $\binom{7}{3}\binom{5}{3}+\binom{7}{2}\binom{5}{3}+\binom{7}{3}\binom{5}{2}=910$ possible committees. There are $\binom{7}{3}\binom{5}{3}$ in which neither feuding party serves; $\binom{7}{2}\binom{5}{3}$ in which the feuding women serves; and $\binom{7}{3}\binom{5}{2}$ in which the feuding man serves.

Exercise 43.

A person has 8 friends, of whom 5 will be invited to a party.

(a) *How many choices are there if 2 of the friends are feuding and will not attend together?*

(b) *How many choices if 2 of the friends will only attend together?*

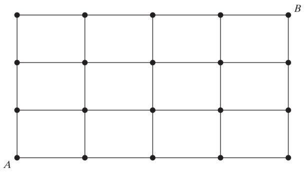
Solution:

(a) $\binom{6}{5} + \binom{2}{1}\binom{6}{4}$

(b) $\binom{6}{5} + \binom{6}{3}$

Exercise 44.

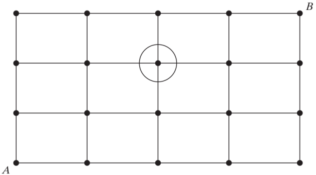
Consider the grid of points shown here. Suppose that, starting at the point labeled A, you can go one step up or one step to the right at each move. This procedure is continued until the point labeled B is reached. How many different paths from A to B are possible?



Solution: $\frac{7!}{3!4!} = 35$. Each path is a linear arrangement of 4 r's and 3 u's (r for right and u for up). For instance the arrangement r, r, u, u, r, r, u specifies the path whose first 2 steps are to the right, next 2 steps are up, next 2 are to the right, and final step is up.

Exercise 45.

Consider the grid of points shown here. Suppose that, starting at the point labeled A, you can go one step up or one step to the right at each move. This procedure is continued until the point labeled B is reached. How many different paths are there from A to B that go through the point circled in the following lattice?



Solution: There are $\frac{4!}{2!2!}$ paths from A to the circled point; and $\frac{3!}{2!1!}$ paths from the circled point to B. Thus, by the basic principle, there are 18 different paths from A to B that go through the circled point.

Exercise 46.

Expand $(3x^2 + y)^5$.

Exercise 47.

The game of bridge is played by 4 players, each of whom is dealt 13 cards. How many bridge deals are possible?

Solution: $\binom{52}{13, 13, 13, 13}$

Exercise 48.

Expand $(x_1 + 2x_2 + 3x_3)^4$.

Exercise 49.

If 12 *people are to be divided into* 3 *committees of respective sizes* $3, 4,$ *and* 5*, how many divisions are possible?*

Solution: $\binom{12}{3,4,5} = \frac{12!}{3!4!5!}$

Exercise 50.

If 8 new teachers are to be divided among 4 schools, how many divisions are possible? What if each school must receive 2 teachers?

Solution: Assuming teachers are distinct.

(a) 4^8

(b) $\binom{8}{2,2,2,2} = \frac{8!}{(2)^4} = 2520.$

Exercise 51.

If 8 identical blackboards are to be divided among 4 schools, how many divisions are possible? How many, if each school must receive at least 1 blackboard?

Solution:

(a) Number of non-negative integer solutions of $x_1 + x_2 + x_3 + x_4 = 8$. Hence, answer is $\binom{11}{3} = 165$.

(b) Here it is the number of positive solutions, hence answer is $\binom{7}{3} = 35$.

Exercise 52.

An elevator starts at the basement with 8 people (not including the elevator operator) and discharges them all by the time it reaches the top floor, number 6. In how many ways could the operator have perceived the people leaving the elevator if all people look alike to him? What if the 8 people consisted of 5 men and 3 women and the operator could tell a man from a woman?

Solution:

(a) Number of non-negative solutions of $x_1 + \ldots + x_6 = 8$. Answer is $= \binom{13}{5}$

(b) (number of solutions of $x_1 + \ldots + x_6 = 5$)$\times$(number of solutions of $x_1 + \ldots + x_6 = 3$) $= \binom{10}{5}\binom{8}{5}$.

Exercise 53.

We have 20 thousand dollars that must be invested among 4 possible opportunities. Each investment must be integral in units of 1 thousand dollars, and there are minimal investments that need to be made if one is to invest in these opportunities. The minimal investments are 2, 2, 3, and 4 thousand dollars. How many different investment strategies are available if

(a) *an investment must be made in each opportunity?*

(b) *investments must be made in at least 3 of the 4 opportunities?*

Solution:

(a) $x_1 + x_2 + x_3 + x_4 = 20,\ x_1 \geq 2,\ x_2 \geq 2,\ x_3 \geq 3,\ x_4 \geq 4.$
Let $y_1 = x_1 - 1,\ y_2 = x_2 - 1,\ y_3 = x_3 - 2,\ y_4 = x_4 - 3$

$$y_1 + y_2 + y_3 + y_4 = 13, y_i > 0.$$

Hence, there are $\binom{12}{3} = 220$ possible strategies.

Solution (contd...)

(b) There are $\binom{15}{2}$ investments only in $1, 2, 3$.

There are $\binom{14}{2}$ investments only in $1, 2, 4$.

There are $\binom{13}{2}$ investments only in $1, 3, 4$.

There are $\binom{13}{2}$ investments only in $2, 3, 4$.

$\binom{15}{2} + \binom{14}{2} + 2\binom{13}{2} + \binom{12}{3} = 552$ possibilities.

Exercise 54.

How many vectors $x_1, \ldots, x_k$ *are there for which each* x_i *is a positive integer such that* $1 \leq x_i \leq n$ *and* $x_1 < x_2 < \cdots < x_k$*?*

Solution: $\binom{n}{k}$

Exercise 55.

Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$

Hint: Consider a group of n men and m women. How many groups of size r are possible?

Solution: There are $\binom{n+m}{r}$ group of size r. As there are $\binom{n}{i}\binom{m}{r-i}$ groups of size r that consist of i men and $r-i$ women, we see that

$$\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}.$$

Exercise 56.

The following identity is known as Fermat's combinatorial identity:

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1} \quad n \geq k.$$

Give a combinatorial argument (no computations are needed) to establish this identity.

Solution: The number of subsets of size k that have i as their highest numbered member is equal to $\binom{i-1}{k-1}$, the number of ways of choosing $k-1$ of the numbers $1, \ldots, i-1$. Summing over i yields the number of subsets of size k.

Exercise 57.

Consider the following combinatorial identity:

$$\sum_{k=1}^{n} k \binom{n}{k} = n \cdot 2^{n-1}.$$

(a) *Present a combinatorial argument for this identity by considering a set of n people and determining, in two ways, the number of possible selections of a committee of any size and a chairperson for the committee.*
Hint:
  (i) *How many possible selections are there of a committee of size k and its chairperson?*
  (ii) *How many possible selections are there of a chairperson and the other committee members?*

Exercise (contd...)

(b) Verify the following identity for $n = 1, 2, 3, 4, 5$:

$$\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{n-2} n(n+1).$$

For a combinatorial proof of the preceding, consider a set of n people and argue that both sides of the identity represent the number of different selections of a committee, its chairperson, and its secretary (possibly the same as the chairperson).

Hint:

(i) How many different selections result in the committee containing exactly k people?

(ii) How many different selections are there in which the chairperson and the secretary are the same? (ANSWER: $n2^{n-1}$.)

(iii) How many different selections result in the chairperson and the secretary being different?

c Now argue that

$$\sum_{k=1}^{n} \binom{n}{k} k^3 = 2^{n-3} n^2(n+3).$$

Solution

Solution: Number of possible selections of a committee of size k and a chairperson is $k\binom{n}{k}$ and so $\sum_{k=1}^{n} k\binom{n}{k}$ represents the desired number. On the other hand, the chairperson can be anyone of the n persons and then each of the other $n-1$ can either be on or off the committee. Hence, $n2^{n-1}$ also represents the desired quantity.

(i) $\binom{n}{k}k^2$

(ii) $n2^{n-1}$ since there are n possible choices for the combined chairperson and secretary and then each of the other $n-1$ can either be on or off the committee.

(iii) $n(n-1)2^{n-2}$

Solution (contd...)

(c) From a set of n we want to choose a committee, its chairperson its secretary and its treasurer (possibly the same). The result follows since

(a) there are $n2^{n-1}$ selections in which the chair, secretary and treasurer are the same person.

(b) there are $3n(n-1)2^{n-2}$ selection in which the chair, secretary and treasurer jobs are held by 2 people.

(c) there are $n(n-1)(n-2)2^{n-3}$ selections in which the chair, secretary and treasurer are all different.

(d) there are $\binom{n}{k} k^3$ selections in which the committee is of size k.

Exercise 58.

Show that, for $n > 0$,

$$\sum_{i=0}^{n}(-1)^i \binom{n}{i} = 0.$$

Solution: $(1-1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-1}$

Exercise 59.

In how many ways can n identical balls be distributed into r urns so that the ith urn contains at least m_i balls, for each $i = 1, \ldots, r$? Assume that $n \geq \sum_{i=1}^{r} m_i$.

Solution: The number of integer solutions of

$$x_1 + \cdots + x_r = n, \; x_i \geq m_i$$

is the same as the number of nonnegative solutions of

$$y_1 + \cdots + y_r = n - \sum_{1}^{r} m_i, \;\; y_i \geq 0.$$

Hence it is $\binom{n - \sum_{1}^{r} m_i + r - 1}{r - 1}$.

Exercise 60.

Argue that there are exactly $\binom{r}{k}\binom{n-1}{n-r+k}$ *solutions of*

$$x_1 + x_2 + \cdots + x_r = n$$

for which exactly k *of the* x_i *are equal to* 0.

Solution: There are $\binom{r}{k}$ choices of the k of the x's to equal 0. Given this choice the other $r-k$ of the x's must be positive and sum to n. Hence there are $\binom{n-1}{r-k-1} = \binom{n-1}{n-r+k}$ such solutions. Hence the result follows.

Exercise 61.

Determine the number of vectors $(x_1, \ldots, x_n)$ such that each x_i is a nonnegative integer and

$$\sum_{i=1}^{n} x_i \leq k.$$

Solution: There are $\binom{j+n-1}{j}$ nonnegative integer solutions of

$$\sum_{i=1}^{n} x_i = j.$$

Hence, there are $\sum_{j=0}^{k} \binom{j+n-1}{j}$ such vectors.

Example 62.

If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7 *?*

Answer: $\frac{1}{6}$

Example 63.

If 3 *balls are "randomly drawn" from a bowl containing* 6 *white and* 5 *black balls, what is the probability that one of the balls is white and the other two black?*

Answer : $\frac{4}{11}$

Example 64.

A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

Answer: $\frac{240}{1001}$

Example 65.

An urn contains n balls, one of which is special. If k of these balls are withdrawn one at a time, with each selection being equally likely to be any of the balls that remain at the time, what is the probability that the special ball is chosen?

Answer : $\frac{k}{n}$

Example 66.

Suppose that $n+m$ balls, of which n are red and m are blue, are arranged in a linear order in such a way that all $(n+m)!$ possible orderings are equally likely. If we record the result of this experiment by listing only the colors of the successive balls, show that all the possible results remain equally likely.

Solution : Consider any one of the $(n+m)!$ possible orderings, and note that any permutation of the red balls among themselves and of the blue balls among themselves does not change the sequence of colors. As a result, every ordering of colorings corresponds to $n!\ m!$ different orderings of the $n+m$ balls, so every ordering of the colors has probability $\frac{n!m!}{(n+m)!}$ of occurring.

For example, suppose that there are 2 red balls, numbered r_1, r_2, and 2 blue balls, numbered b_1, b_2. Then, of the 4! possible orderings, there will be 2! 2! orderings that result in any specified color combination. For instance, the following orderings result in the successive balls alternating in color, with a red ball first:

$$r_1, b_1, r_2, b_2 \quad r_1, b_2, r_2, b_1 \quad r_2, b_1, r_1, b_2 \quad r_2, b_2, r_1, b_1$$

Therefore, each of the possible orderings of the colors has probability $\frac{4}{24} = \frac{1}{6}$ of occurring.

Example 67.

A poker hand consists of 5 cards. If the cards have distinct consecutive values and are not all of the same suit, we say that the hand is a straight. For instance, a hand consisting of the five of spades, six of spades, seven of spades, eight of spades, and nine of hearts is a straight. What is the probability that one is dealt a straight?

Answer : $\frac{10(4^5-4)}{\binom{52}{2}} \approx .0039$

Example

Example 68.

A 5-card poker hand is said to be a full house if it consists of 3 cards of the same denomination and 2 other cards of the same denomination (of course, different from the first denomination). Thus, one kind of full house is three of a kind plus a pair. What is the probability that one is dealt a full house?

Answer: $\dfrac{13 \cdot 12 \cdot \binom{4}{2}\binom{4}{3}}{\binom{52}{5}} \approx .0014$

Example 69.

In the game of bridge, the entire deck of 52 cards is dealt out to 4 players. What is the probability that

(a) *one of the players receives all 13 spades;*

(b) *each player receives 1 ace?*

Solution :

(a) $\frac{4}{\binom{52}{13}} \approx 6.3 \times 10^{-12}$

(b) $\frac{4!\binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}} \approx 0.1055$

Example

Some results in probability are quite surprising when initially encountered. Our next two examples illustrate this phenomenon.

Example 70.

If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$?

Example 71.

A deck of 52 playing cards is shuffled, and the cards are turned up one at a time until the first ace appears. Is the next card – that is, the card following the first ace – more likely to be the ace of spades or the two of clubs?

Example

Example 72.

A football team consists of 20 *offensive and* 20 *defensive players. The players are to be paired in groups of* 2 *for the purpose of determining roommates. If the pairing is done at random, what is the probability that there are no offensive–defensive roommate pairs? What is the probability that there are* $2i$ *offensive–defensive roommate pairs,* $i = 1, 2, ..., 10$?

Example

Example 73.

A total of 36 *members of a club play tennis,* 28 *play squash, and* 18 *play badminton. Furthermore,* 22 *of the members play both tennis and squash,* 12 *play both tennis and badminton,* 9 *play both squash and badminton, and* 4 *play all three sports. How many members of this club play at least one of three sports?*

The next example not only possesses the virtue of giving rise to a somewhat surprising answer, but is also of theoretical interest.

Exercises 74.

1. *Out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover.*
2. *A six-sided die is loaded in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. Construct a probabilistic model for a single roll of this die and find the probability that the outcome is less than 4.*
3. *A four-sided die is rolled repeatedly, until the first time (if ever) that an even number is obtained. What is the sample space for this experiment?*
4. *You enter a special kind of chess tournament, in which you play one game with each of three opponents, but you get to choose the order in which you play your opponents, knowing the probability of a win against each. You win the tournament if you win two games in a row, and you want to maximize the probability of winning. Show that it is optimal to play the weakest opponent second, and that the order of playing the other two opponents does not matter.*

Example 75.

Suppose that each of N men at a party throws his hat into the center of the room. The hats are first mixed up, and then each man randomly selects a hat. What is the probability that none of the men selects his own hat?

Example 76.

Compute the probability that 10 married couples are seated at random at a round table, then no wife sits next to her husband.

Probability as a continuous set function

A sequence of events $\{E_n, n \geq 1\}$ is said to be an increasing sequence if

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset E_{n+1} \subset \cdots$$

whereas it is said to be a decreasing sequence if
$E_1 \supset E_2 \supset \cdots \supset E_n \supset E_{n+1} \supset \cdots$

If $\{E_n, n \geq 1\}$ is an increasing sequence of events, then we define a new event, denoted by $\lim_{n\to\infty} E_n$, by

$$\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$$

Similarly, if $\{E_n, n \geq 1\}$ is a decreasing sequence of events, we define $\lim_n E_n$ by

$$\lim_{n\to\infty} E_n = \bigcap_{i=1}^{\infty} E_i$$

Theorem 77.

If $\{E_n, n \geq 1\}$ is either an increasing or a decreasing sequence of events, then

$$\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n).$$

Probability and a paradox

Example 78.

Suppose that we possess an infinitely large urn and an infinite collection of balls labeled ball number 1*, number* 2*, number* 3*, and so on. Consider an experiment performed as follows: At* 1 *minute to* 12 *P.M., balls numbered* 1 *through* 10 *are placed in the urn and ball number 10 is withdrawn. (Assume that the withdrawal takes no time.) At* $\frac{1}{2}$ *minute to* 12 *P.M., balls numbered* 11 *through* 20 *are placed in the urn and ball number* 20 *is withdrawn. At* $\frac{1}{4}$ *minute to* 12 *P.M., balls numbered* 21 *through* 30 *are placed in the urn and ball number* 30 *is withdrawn. At* $\frac{1}{8}$ *minute to* 12 *P.M., and so on. The question of interest is, How many balls are in the urn at* 12 *P.M.?*

The answer to this question is clearly that there is an infinite number of balls in the urn at 12 P.M., since any ball whose number is not of the form $10n$, $n \geq 1$, will have been placed in the urn and will not have been withdrawn before 12 P.M. Hence, the problem is solved when the experiment is performed as described.

However, let us now change the experiment and suppose that at 1 minute to 12 P.M. balls numbered 1 through 10 are placed in the urn and ball number 1 is withdrawn; at $\frac{1}{2}$ minute to 12 P.M., balls numbered 11 through 20 are placed in the urn and ball number 2 is withdrawn; at $\frac{1}{4}$ minute to 12 P.M., balls numbered 21 through 30 are placed in the urn and ball number 3 is withdrawn; at $\frac{1}{8}$ minute to 12 P.M., balls numbered 31 through 40 are placed in the urn and ball number 4 is withdrawn, and so on. For this new experiment, how many balls are in the urn at 12 P.M.?

Probability and a paradox

Surprisingly enough, the answer now is that the urn is empty at 12 P.M. For, consider any ball – say, ball number n. At some time prior to 12 P.M. [in particular, at $(\frac{1}{2})^{n-1}$ minutes to 12 P.M.], this ball would have been withdrawn from the urn. Hence, for each n, ball number n is not in the urn at 12 P.M.; therefore, the urn must be empty at that time.

Probability and a paradox

We see then, from the preceding discussion that the manner in which the balls are withdrawn makes a difference. For, in the first case only balls numbered $10n, n \geq 1$, are ever withdrawn, whereas in the second case all of the balls are eventually withdrawn. Let us now suppose that whenever a ball is to be withdrawn, that ball is randomly selected from among those present. That is, suppose that at 1 minute to 12 P.M. balls numbered 1 through 10 are placed in the urn and a ball is randomly selected and withdrawn, and so on. In this case, how many balls are in the urn at 12 P.M.?

Probability as a measure of belief

Thus far we have interpreted the probability of an event of a given experiment as being a measure of how frequently the event will occur when the experiment is continually repeated.

However, there are also other uses of the term *probability*. For instance, we have all heard such statements as “It is 90 percent probable that Shakespeare actually wrote *Hamlet*” or “The probability that Oswald acted alone in assassinating Kennedy is .8.” How are we to interpret these statements?

Probability as a measure of belief

The most simple and natural interpretation is that the probabilities referred to are measures of the individual's degree of belief in the statements that he or she is making. In other words, the individual making the foregoing statements is quite certain that Oswald acted alone and is even more certain that Shakespeare wrote Hamlet. This interpretation of probability as being a measure of the degree of one's belief is often referred to as the *personal or subjective* view of probability.

It seems logical to suppose that a "measure of the degree of one's belief" should satisfy all of the axioms of probability. For example, if we are 70 percent certain that Shakespeare wrote *Julius Caesar* and 10 percent certain that it was actually Marlowe, then it is logical to suppose that we are 80 percent certain that it was either Shakespeare or Marlowe. Hence, whether we interpret probability as a measure of belief or as a long-run frequency of occurrence, its mathematical properties remain unchanged.

Example 79.

Suppose that, in a 7-horse race, you feel that each of the first 2 *horses has a* 20 *percent chance of winning, horses* 43 *and* 4 *each have a* 15 *percent chance, and the remaining* 3 *horses have a* 10 *percent chance each. Would it be better for you to wager at even money that the winner will be one of the first three horses or to wager, again at even money, that the winner will be one of the horses* $1, 5, 6,$ *and* 7?

Solution : On the basis of your personal probabilities concerning the outcome of the race, your probability of winning the first bet is $.2 + .2 + .15 = .55$, whereas it is $.2 + .1 + .1 + .1 = .5$ for the second bet. Hence, the first wager is more attractive.

Example

Note that, in supposing that a person's subjective probabilities are always consistent with the axioms of probability, we are dealing with an idealized rather than an actual person. For instance, if we were to ask someone what he thought the chances were of

(a) rain today,
(b) rain tomorrow,
(c) rain both today and tomorrow,
(d) rain either today or tomorrow,

it is quite possible that, after some deliberation, he might give 30 percent, 40 percent, 20 percent, and 60 percent as answers. Unfortunately, such answers (or such subjective probabilities) are not consistent with the axioms of probability. (Why not?) We would of course hope that, after this was pointed out to the respondent, she would change his answers. (One possibility we could accept is 30 percent, 40 percent, 10 percent, and 60 percent.)

Summary : $P(A)$ can be interpreted either as a long-run relative frequency or as a measure of one's degree of belief.

Exercise 80.

A box contains 3 marbles: 1 red, 1 green, and 1 blue. Consider an experiment that consists of taking 1 marble from the box and then replacing it in the box and drawing a second marble from the box. Describe the sample space. Repeat when the second marble is drawn without replacing the first marble.

Solution:

(a) $S = \{(r,r),(r,g),(r,b),(g,r),(g,g),(g,b),(b,r),(b,g),(b,b)\}$
(b) $S = \{(r,g),(r,b),(g,r),(g,b),(b,r),(b,g)\}$

Exercise 81.

In an experiment, die is rolled continually until a 6 appears, at which point the experiment stops. What is the sample space of this experiment? Let E_n denote the event that n rolls are necessary to complete the experiment. What points of the sample space are contained in E_n? What is $\left(\bigcup_{1}^{\infty} E_n\right)^c$?

Solution: $S = \{(n, x_1, \ldots, x_{n-1}),\ n \geq 1,\ x_i \neq 6,\ i = 1, \ldots, n-1\}$, with the interpretation that the outcome is $(n, x_1, \ldots, x_{n-1})$ if the first 6 appears on roll n, and x_i appears on roll, i, $i = 1, \ldots, n-1$. The event $(\cup_{n=1}^{\infty} E_n)^c$ is the event that 6 never appears.

Exercise 82.

A system is comprised of 5 *components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector* $(x_1, x_2, x_3, x_4, x_5)$*, where* x_i *is equal to* 1 *if component* i *is working and is equal to* 0 *if component* i *is failed.*

(a) *How many outcomes are in the sample space of this experiment?*

(b) *Suppose that the system will work if components* 1 *and* 2 *are both working, or if components* 3 *and* 4 *are both working, or if components* $1, 3,$ *and* 5 *are all working. Let* W *be the event that the system will work. Specify all the outcomes in* W*.*

(c) *Let* A *be the event that components* 4 *and* 5 *are both failed. How many outcomes are contained in the event* A*?*

(d) *Write out all the outcomes in the event* AW*.*

Solution (contd...)

Solution:

(a) $2^5 = 32$

(b)

$$W = \{(1,1,1,1,1),(1,1,1,1,0),(1,1,1,0,1),(1,1,0,1,1),(1,1,1,0,0),\\ (1,1,0,1,0),(1,1,0,0,1),(1,1,0,0,0),(1,0,1,1,1),(0,1,1,1,1),\\ (1,0,1,1,0),(0,1,1,1,0),(0,0,1,1,1),(0,0,1,1,0),(1,0,1,0,1)\}$$

(c) 8

(d) $AW = \{(1,1,1,0,0),(1,1,0,0,0)\}$

Exercise 83.

Consider an experiment that consists of determining the type of job-either blue-collar or white-collar-and the political affiliation-Republican, Democratic, or Independent-of the 15 *members of an adult soccer team. How many outcomes are*

(a) *in the sample space?*
(b) *in the event that at least one of the team members is a blue-collar worker?*
(c) *in the event that none of the team members considers himself or herself an Independent?*

Solution:

(a) 6^{15}
(b) $6^{15} - 3^{15}$
(c) 4^{15}

Exercise 84.

Suppose that A and B are mutually exclusive events for which $P(A) = .3$ and $P(B) = .5$. What is the probability that

(a) *either A or B occurs?*

(b) *A occurs but B does not?*

(c) *both A and B occur?*

Solution:

(a) .8

(b) .3

(c) 0

Exercise 85.

A total of 28 percent of American males smoke cigarettes, 7 percent smoke cigars, and 5 percent smoke both cigars and cigarettes.

(a) *What percentage of males smokes neither cigars nor cigarettes?*

(b) *What percentage smokes cigars but not cigarettes?*

Solution: Let A be the event that a randomly chosen person is a cigarette smoker and let B be the event that she or he is a cigar smoker.

(a) $1 - P(A \cup B) = 1 - (.07 + .28 - .05) = .7$. Hence, 70 percent smoke neither.

(b) $P(A^c B) = P(B) - P(AB) = .07 - .05 = .02$. Hence, 2 percent smoke cigars but not cigarettes.

Exercise 86.

An elementary school is offering 3 *language classes: one in Spanish, one in French, and one in German. The classes are open to any of the* 100 *students in the school. There are* 28 *students in the Spanish class,* 26 *in the French class, and* 16 *in the German class. There are* 12 *students that are in both Spanish and French,* 4 *that are in both Spanish and German, and* 6 *that are in both French and German. In addition, there are* 2 *students taking all* 3 *classes.*

(a) *If a student is chosen randomly, what is the probability that he or she is not in any of the language classes?*

(b) *If a student is chosen randomly, what is the probability that he or she is taking exactly one language class?*

(c) *If* 2 *students are chosen randomly, what is the probability that at least* 1 *is taking a language class?*

Solution (contd...)

Solution:

(a) $P(S \cup F \cup G) = (28 + 26 + 16 - 12 - 4 - 6 + 2)/100 = 1/2$
The desired probability is $1 - 1/2 = 1/2$.

(b) Use the Venn diagram below to obtain the answer $32/100$.

(c) since 50 students are not taking any of the courses, the probability that neither one is taking a course is $\binom{50}{2} \Big/ \binom{100}{2} = 49/198$ and so the probability that at least one is taking a course is $149/198$.

Exercise 87.

If it is assumed that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt

(a) *a flush? (A hand is said to be a flush if all 5 cards are of the same suit.)*

(b) *one pair? (This occurs when the cards have denominations a, a, b, c, d, where a, b, c, and d are all distinct.)*

(c) *two pairs? (This occurs when the cards have denominations a, a, b, b, c, where a, b, and c are all distinct.)*

(d) *three of a kind? (This occurs when the cards have denominations a, a, a, b, c, where a, b, and c are all distinct.)*

(e) *four of a kind? (This occurs when the cards have denominations a, a, a, a, b.)*

Solution (contd...)

Solution:

(a) $4\binom{13}{5} \Big/ \binom{52}{5}$

(b) $13\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1} \Big/ \binom{52}{5}$

(c) $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1} \Big/ \binom{52}{5}$

(d) $13\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1} \Big/ \binom{52}{5}$

(e) $13\binom{4}{4}\binom{48}{1} \Big/ \binom{52}{5}$

Exercise 88.

Poker dice is played by simultaneously rolling 5 *dice. Show that*

(a) $P\{\textit{no two alike}\} = .0926;$

(b) $P\{\textit{one pair}\} = .4630;$

(c) $P\{\textit{two pair}\} = .2315;$

(d) $P\{\textit{three alike}\} = .1543;$

(e) $P\{\textit{full house}\} = .0386;$

(f) $P\{\textit{four alike}\} = .0193;$

(g) $P\{\textit{five alike}\} = .0008.$

Solution (contd...)

Solution:

(a) $\frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{6^5}$

(b) $\frac{6\binom{5}{2}5\cdot 4\cdot 3}{6^5}$

(c) $\frac{\binom{6}{2}4\binom{5}{2}\binom{3}{2}}{6^5}$

(d) $\frac{6\cdot 5\cdot 4\binom{5}{3}}{21}$

(e) $\frac{6\cdot 5\binom{5}{3}}{6^5}$

(f) $\frac{6\cdot 5\binom{5}{4}}{6^5}$

(g) $\frac{6}{6^5}$

Exercise 89.

Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?

Solution: $\frac{2 \cdot 4 \cdot 16}{52 \cdot 51}$

Exercise 90.

Two symmetric dice have both had two of their sides painted red, two painted black, one painted yellow, and the other painted white. When this pair of dice is rolled, what is the probability that both dice land with the same color face up?

Solution: $4/36 + 4/36 + 1/36 + 1/36 = 5/18$

Exercise 91.

A pair of fair dice is rolled. What is the probability that the second die lands on a higher value than does the first?

Solution: The answer is $5/12$, which can be seen as follows:
$1 = P\{\text{first higher}\} + P\{\text{second higher}\} + p\{\text{same}\}$
$= 2P\{\text{second higher}\} + p\{\text{same}\}$
$= 2P\{\text{second higher}\} + 1/6$
Another way of solving is to list all the outcomes for which the second is higher. There is 1 outcome when the second die lands on two, 2 when it lands on three, 3 when it lands on four, 4 when it lands on five, and 5 when it lands on six. Hence, the probability is $(1 + 2 + 3 + 4 + 5)/36 = 5/12$.

Exercise 92.

If two dice are rolled, what is the probability that the sum of the upturned faces equals i? Find it for $i = 2, 3, \ldots, 11, 12$.

Solution:

Exercise 93.

A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first. Hint: Let E_n denote the event that a 5 occurs on the nth roll and no 5 or 7 occurs on the first $n-1$ rolls. Compute $P(E_n)$ and argue that $\sum_{n=1}^{\infty} P(E_n)$ is the desired probability.

Solution: $P(E_n) = \left(\frac{26}{36}\right)^{n-1} \frac{6}{36}$, $\sum_{n=1}^{\infty} P(E_n) = \frac{2}{5}$

Exercise 94.

An urn contains 3 red and 7 black balls. Players A and B withdraw balls from the urn consecutively until a red ball is selected. Find the probability that A selects the red ball. (A draws the first ball, then B, and so on. There is no replacement of the balls drawn.)

Solution: Imagine that all 10 balls are withdrawn

$$P(A) = \frac{3 \cdot 9! + 7 \cdot 6 \cdot 3 \cdot 7! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 5! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 3 \cdot 3!}{10!}$$

Exercise 95.

An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be (a) of the same color? (b) of different colors? Repeat under the assumption that whenever a ball is selected, its color is noted and it is then replaced in the urn before the next selection. This is known as sampling with replacement.

Solution: $P\{\text{same}\} = \dfrac{\binom{5}{3}+\binom{6}{3}+\binom{8}{3}}{\binom{19}{3}}$

$P\{\text{different}\} = \binom{5}{1}\binom{6}{1}\binom{8}{1} \Big/ \binom{19}{3}$

If sampling is with replacement

$P\{\text{same}\} = \frac{5^3+6^3+8^3}{(19)^3}$

$P\{\textit{different}\} = P(\text{RBG}) + P(\text{BRG}) + P(\text{RGB}) + \ldots + P(\text{GBR})$
$= \frac{6 \cdot 5 \cdot 6 \cdot 8}{(19)^3}$

Exercise 96.

An urn contains n white and m black balls, where n and m are positive numbers.

(a) *If two balls are randomly withdrawn, what is the probability that they are the same color?*

(b) *If a ball is randomly withdrawn and then replaced before the second one is drawn, what is the probability that the withdrawn balls are the same color?*

(c) *Show that the probability in part (b) is always larger than the one in part (a).*

Solution:

(a) $\frac{n(n-1)+m(m-1)}{(n+m)(n+m-1)}$

(b) Putting all terms over the common denominator $(n+m)^2(n+m-1)$ shows that we must prove that

$$n^2(n+m-1)+m^2(n+m-1)\geq n(n-1)(n+m)+m(m-1)(n+m)$$

which is immediate upon multiplying through and simplifying.

Exercise 97.

A group of individuals containing b boys and g girls is lined up in random order; that is, each of the $(b+g)!$ permutations is assumed to be equally likely. What is the probability that the person in the ith position, $1 \leq i \leq b+g$, is a girl?

Solution: $\frac{g(b+g-1)!}{(b+g)!} = \frac{g}{b+g}$

Exercise 98.

Seven balls are randomly withdrawn from an urn that contains 12 red, 16 blue, and 18 green balls. Find the probability that

(a) *3 red, 2 blue, and 2 green balls are withdrawn;*

(b) *at least 2 red balls are withdrawn;*

(c) *all withdrawn balls are the same color;*

(d) *either exactly 3 red balls or exactly 3 blue balls are withdrawn.*

Solution: $1 - \binom{30}{3} \Big/ \binom{54}{3} \approx .8363$

Exercise 99.

There are n socks, 3 of which are red, in a drawer. What is the value of n if, when 2 of the socks are chosen randomly, the probability that they are both red is $\frac{1}{2}$?

Solution: $1/2 = \binom{3}{2} \Big/ \binom{n}{2}$ or $n(n-1) = 12$ or $n = 4$.

Exercise 100.

There are 5 hotels in a certain town. If 3 people check into hotels in a day, what is the probability that they each check into a different hotel? What assumptions are you making?

Solution: $\frac{5 \cdot 4 \cdot 3}{5 \cdot 5 \cdot 5} = \frac{12}{25}$

Exercise 101.

If a die is rolled 4 times, what is the probability that 6 comes up at least once?

Solution: $1 - \frac{5^4}{6^4}$

Exercise 102.

Two dice are thrown n times in succession. Compute the probability that double 6 appears at least once. How large need n be to make this probability at least $\frac{1}{2}$?

Solution: $1 - \left(\frac{35}{36}\right)^n$

Exercise 103.

(a) *If N people, including A and B, are randomly arranged in a line, what is the probability that A and B are next to each other?*

(b) *What would the probability be if the people were randomly arranged in a circle?*

Solution: $\frac{2(n-1)(n-2)}{n!} = \frac{2}{n}$ in a line
$\frac{2n(n-2)!}{n!} = \frac{2}{n-1}$ if in a circle, $n \geq 2$

Exercise 104.

A woman has n keys, of which one will open her door.

(a) *If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her kth try?*

(b) *What if she does not discard previously tried keys?*

Solution: $1/n$ if discard, $\frac{(n-1)^{k-1}}{n^k}$ if do not discard

Exercise 105.

Given 20 people, what is the probability that, among the 12 months in the year, there are 4 months containing exactly 2 birthdays and 4 containing exactly 3 birthdays?

Solution: $\binom{12}{4}\binom{8}{4}\frac{(20)!}{(3!)^4(2!)^4}\Big/(12)^{20}$

Exercise 106.

A group of 6 men and 6 women is randomly divided into 2 groups of size 6 each. What is the probability that both groups will have the same number of men?

Solution: $\binom{6}{3}\binom{6}{3} \Big/ \binom{12}{6}$

Exercise 107.

A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be

1. *no complete pair?*
2. *exactly* 1 *complete pair?*

Solution:

1. $\frac{20 \cdot 18 \cdot 16 \cdot 14 \cdot 12 \cdot 10 \cdot 8 \cdot 6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}$
2. $\frac{\binom{10}{1}\binom{9}{6}\frac{8!}{2!}2^6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}$

Exercise 108.

If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.

Solution: Let A_i be the event that couple i sit next to each other. Then

$$P\left(\cup_{i=1}^{4} A_i\right) = 4\frac{2 \cdot 7!}{8!} - 6\frac{2^2 \cdot 6!}{8!} + 4\frac{2^3 \cdot 5!}{8!} - \frac{2^4 \cdot 4!}{8!}$$

and the desired probability is 1 minus the preceding.

Exercise

Notation : We say EF for $E \cap F$ when E and F are two events.

Exercise 109.

Prove the following relations

1. $EF \subset E \subset E \cup F$.
2. *If* $E \subset F$, *then* $F^c \subset E^c$.
3. $F = FE \cup FE^c$ *and* $E \cup F = E \cup E^c F$.
4. $\left(\bigcup_1^\infty E_i \right) F = \bigcup_1^\infty E_i F$ *and* $\left(\bigcap_1^\infty E_i \right) \cup F = \bigcap_1^\infty (E_i \cup F)$.

Exercise 110.

For any sequence of events $E_1, E_2, \ldots,$ *define a new sequence* $F_1, F_2, \ldots$ *of disjoint events (that is, events such that* $F_iF_j = \emptyset$ *whenever* $i \neq j$ *) such that for all* $n \geq 1$,

$$\bigcup_1^n F_i = \bigcup_1^n E_i$$

Solution:

$$F_i = E_i \cap_{j=1}^{i=1} E_j^c$$

Exercise 111.

Let E, F, and G be three events. Find expressions for the events so that, of E, F, and G,

(a) *only E occurs;*

(b) *both E and G, but not F, occur;*

(c) *at least one of the events occurs;*

(d) *at least two of the events occur;*

(e) *all three events occur;*

(f) *none of the events occurs;*

(g) *at most one of the events occurs;*

(h) *at most two of the events occur;*

(i) *exactly two of the events occur;*

(j) *at most three of the events occur.*

Solution

Solution:

(a) EF^cG^c

(b) EF^cG

(c) $E \cup F \cup G$

(d) $EF \cup EG \cup FG$

(e) EFG

(f) $E^cF^cG^c$

(g) $E^cF^cG^c \cup EF^cG^c \cup E^cFG^c \cup E^cF^cG$

(h) $(EFG)^c$

(i) $EFG^c \cup EF^cG \cup E^cFG$

(j) S

Exercise 112.

Use induction to generalize Bonferroni's inequality to n events. That is, show that

$$P(E_1 E_2 \ldots E_n) \geq P(E_1) + \ldots + P(E_n) - (n-1)$$

Solution:

$P(E_1 \ldots E_n) \geq P(E_1 \ldots E_{n-1}) + P(E_n) - 1$ by Bonferonni's Inequality.

$\geq \sum_{1}^{n-1} P(E_i) - (n-2) + P(E_n) - 1$ by induction hypothesis.

Exercises 113.

1. *Let f_n denote the number of ways of tossing a coin n times such that successive heads never appear. Argue that*

$$f_n = f_{n-1} + f_{n-2} \quad n \geq 2, \text{ where } f_0 = 1, f_1 = 2$$

   Hint: How many outcomes are there that start with a head, and how many start with a tail? If P_n denotes the probability that successive heads never appear when a coin is tossed n times, find P_n (in terms of f_n) when all possible outcomes of the n tosses are assumed equally likely. Compute P_{10}.

2. *Consider an experiment whose sample space consists of a countably infinite number of points. Show that not all points can be equally likely. Can all points have a positive probability of occurring?*

Exercises 114.

1. *In a state lottery, a player must choose 8 of the numbers from 1 to 40. The lottery commission then performs an experiment that selects 8 of these 40 numbers. Assuming that the choice of the lottery commission is equally likely to be any of the* $\binom{40}{8}$ *combinations, what is the probability that a player has*
   (a) *all 8 of the numbers selected by the lottery commission?*
   (b) *7 of the numbers selected by the lottery commission?*
   (c) *at least 6 of the numbers selected by the lottery commission?*
2. *Consider an experiment that consists of six horses, numbered 1 through 6, running a race, and suppose that the sample space consists of the 6! possible orders in which the horses finish. Let A be the event that the number-1 horse is among the top three finishers, and let B be the event that the number-2 horse comes in second. How many outcomes are in the event* $A \cup B$?

Exercises 115.

1. *Show that if* $P(A_i) = 1$ *for all* $i \geq 1$*, then* $P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1$.
2. *Let* $T_k(n)$ *denote the number of partitions of the set* $\{1, \ldots, n\}$ *into* k *nonempty subsets, where* $1 \leq k \leq n$*. Argue that*

$$T_k(n) = kT_k(n-1) + T_{k-1}(n-1).$$

   Hint: In how many partitions is $\{1\}$ *a subset, and in how many is* 1 *an element of a subset that contains other elements?*
3. *Balls are randomly removed from an urn initially containing* 20 *red and* 10 *blue balls. What is the probability that all of the red balls are removed before all of the blue ones have been removed?*

References

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